Shell Capacities from Four Spatial Directions

The Periods of the Periodic Table as Shells of One Four-Direction Degree

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Abstract. An electron orbit in four spatial directions keeps a fixed distance from the nucleus, so its shell is a three-sphere. The independent angular patterns of degree m on a three-sphere number exactly $(m+1)^2$. With two electrons per pattern the shell capacities are $2, 8, 18, 32, 50$. Under rotations of the three observed directions, degree m splits into the angular momenta $\ell = 0, 1, \ldots, m$, each exactly once, so a shell of one degree holds every subshell $s, p, d, \ldots$ up to $\ell = m$ once. Reading the degree of each period from its length, $m = 0, 1, 1, 2, 2, 3, 3$, every one of the seven periods of the periodic table has exactly this content. The counting is exact, and the paper gives every step needed to reproduce it.

1. The shell

The electron's orbit runs through four spatial directions [1]. On its locked path

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 \tag{1}$$

at every moment: the orbit never changes its distance from the centre. A shell of such orbits is therefore the three-sphere S^3 of radius r, the set of points at that distance in four directions. In three directions the corresponding shell is the two-sphere S^2.

The angular patterns available on a shell are the harmonic polynomials restricted to it: homogeneous polynomials $P(x)$ of degree m with $\nabla^2 P = 0$. On the unit sphere S^{n-1} in n directions such a pattern is an eigenfunction of the angular Laplacian with eigenvalue $-m(m+n-2)$. In three directions the degree m is the familiar angular momentum ℓ.

2. Counting the patterns

The homogeneous polynomials of degree m in n variables number

$$\dim \mathcal{P}_m^{(n)} = \binom{m+n-1}{n-1}. \tag{2}$$

The Laplacian maps $\mathcal{P}_m^{(n)}$ onto $\mathcal{P}_{m-2}^{(n)}$, and the harmonic polynomials are its kernel, so

$$N_n(m) = \binom{m+n-1}{n-1} - \binom{m+n-3}{n-1}. \tag{3}$$

Here $\binom{a}{b} = 0$ for $a < b$, so for $m = 0$ and 1 the subtracted term vanishes. A homogeneous polynomial is fixed by its values on the sphere, so the patterns on the shell number the same.

In three directions this gives $\binom{m+2}{2} - \binom{m}{2} = 2m+1$, the count $2\ell+1$ of each subshell. In four directions

$$\begin{aligned} N_4(m) &= \binom{m+3}{3} - \binom{m+1}{3} = \frac{(m+1)\big[(m+3)(m+2) - m(m-1)\big]}{6} \\ &= \frac{(m+1)(6m+6)}{6} = (m+1)^2. \end{aligned} \tag{4}$$

Table 1 sets out the terms.

Degree m	$\dim \mathcal{P}_m^{(4)}$	$\dim \mathcal{P}_{m-2}^{(4)}$	Patterns on S^3	Patterns on S^2, $2m+1$	Capacity $2(m+1)^2$
0	1	0	1	1	2
1	4	0	4	3	8
2	10	1	9	5	18
3	20	4	16	7	32
4	35	10	25	9	50

Table 1. Harmonic patterns of degree m from equation (3). The four-direction count is a perfect square at every degree.

3. How a degree splits in the observed directions

Write a point of the unit three-sphere with one angle χ measured from the fourth direction,

$$(x_1, x_2, x_3, x_4) = (\sin\chi\, \hat{n},\ \cos\chi), \qquad 0 \le \chi \le \pi, \tag{5}$$

where $\hat{n}(\theta, \phi)$ is a unit vector in the three observed directions. Rotations of the observed directions turn $\hat{n}$ and leave χ fixed. A complete set of degree-m patterns is

$$Y_{m\ell\mu}(\chi, \theta, \phi) = \sin^\ell \chi\, C_{m-\ell}^{(\ell+1)}(\cos\chi)\, Y_{\ell\mu}(\theta, \phi), \qquad \ell = 0, 1, \ldots, m, \quad \mu = -\ell, \ldots, \ell, \tag{6}$$

with $C_k^{(\lambda)}$ the Gegenbauer polynomials and $Y_{\ell\mu}$ the ordinary spherical harmonics. Each $Y_{m\ell\mu}$ is an eigenfunction of the three-sphere's angular Laplacian with eigenvalue $-m(m+2)$, and under rotations of the observed directions it carries angular momentum ℓ. Every ℓ from 0 to m occurs once, and the count closes exactly:

$$\sum_{\ell=0}^{m} (2\ell+1) = (m+1)^2. \tag{7}$$

The first two degrees can be written out by hand. At $m = 1$ the patterns are x_4 ($\ell = 0$) and x_1, x_2, x_3 ($\ell = 1$): one s and three p. At $m = 2$ they are

$$3x_4^2 - (x_1^2 + x_2^2 + x_3^2)\ (\ell = 0), \qquad x_4 x_i\ (\ell = 1), \qquad x_i x_j - \tfrac{1}{3}\delta_{ij}(x_1^2 + x_2^2 + x_3^2)\ (\ell = 2), \tag{8}$$

with $i, j \in \{1, 2, 3\}$: one s, three p and five d, nine in all, written in the real forms of the same span. Each is harmonic in four directions; for the first, $\nabla^2 = 6 - 6 = 0$.

Hydrogen's level n holds n^2 states, which is the degree $m = n - 1$ of equation (4), split into $\ell = 0, \ldots, n-1$ as in equation (7).

4. Shell capacities and the periods

Two electrons occupy each pattern. A shell of degree m therefore holds

$$2(m+1)^2 = 2,\ 8,\ 18,\ 32,\ 50,\ \ldots \tag{9}$$

and consists of the subshells $\ell = 0, \ldots, m$ once each.

The periods of the periodic table have lengths $2, 8, 8, 18, 18, 32, 32$. Each is $2(m+1)^2$ for one whole number m, and reading m from the length gives the sequence $m = 0, 1, 1, 2, 2, 3, 3$. The content of each period then follows from equation (7) and can be compared with the subshells that fill each period, period by period (Table 2, Figure 1).

Period	Subshells (Madelung order)	ℓ present	Length	m	$2(m+1)^2$
1	1s	0	2	0	2
2	2s 2p	0, 1	8	1	8
3	3s 3p	0, 1	8	1	8
4	4s 3d 4p	0, 1, 2	18	2	18
5	5s 4d 5p	0, 1, 2	18	2	18
6	6s 4f 5d 6p	0, 1, 2, 3	32	3	32
7	7s 5f 6d 7p	0, 1, 2, 3	32	3	32
8	8s 5g 6f 7d 8p	0, 1, 2, 3, 4	50	4	50

Table 2. Each period holds the subshells $\ell = 0, \ldots, m$ exactly once, the branching of one four-direction degree. The eighth row is the next degree.

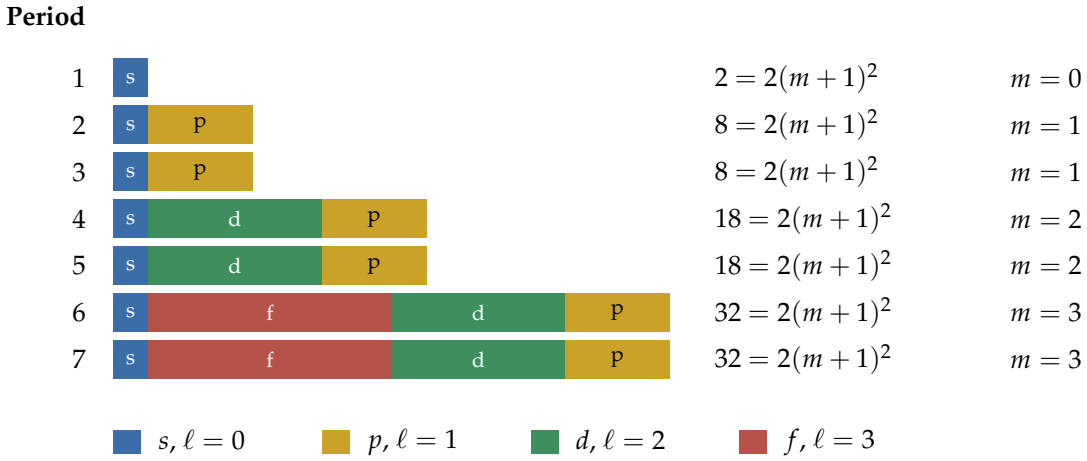


Figure 1. The seven periods drawn as blocks, two electrons per pattern. Each row contains every subshell from s up to $\ell = m$ exactly once, and its length is the four-direction shell capacity of that degree.

The lengths fix the degrees, and the subshell content is then an independent check. Period 4, for example, has 18 places, and 18 could have been filled by other combinations of subshells; the observed $s + d + p$ is exactly the branching of degree 2. The same holds for every period. The next degree, $m = 4$, gives an eighth period of 50 places containing s, p, d, f and g once each.

5. Relation to earlier work

Fock showed in 1935 that hydrogen's bound states in momentum space are harmonics on a three-sphere and that its n^2 degeneracy is the rotation symmetry of that sphere [2]. Rumer and Fet built a description of the periodic system on the group Spin(4) [3], and the group SO(4,2) has since been used for the same purpose [4]. The counting of Sections 2 and 3 is the same mathematics. Here the three-sphere is the shell of the electron's orbit in position space, a real orbit through a fourth spatial direction [1], so the square counts of the shells and of hydrogen's levels come directly from the number of directions the orbit uses.

References

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[4] M. R. Kibler, On the use of the group SO(4,2) in atomic and molecular physics, *Mol. Phys.* **102**, 1221 (2004).